Geometric aspects of supersymmetry and quantisation of fermions

1983 J. Phys. A: Math. Gen. 16 2633

(http://iopscience.iop.org/0305-4470/16/12/012)

J. Phys. A: Math. Gen. 16 (1983) 2633–2639. Printed in Great Britain

Geometric aspects of supersymmetry and quantisation of fermions

Víctor Aldaya and José A de Azcárraga
Departamento de Física Teórica, Facultad de Ciencias Físicas, Universidad de Valencia, Burjasot (Valencia), Spain

Received 1 December 1982

Abstract. We analyse in this paper the geometric content of some aspects of supersymmetry by generalising a procedure of geometric quantisation recently proposed. The method is entirely based on a group, the 'quantum group'; by generalising it to a supergroup we are able to quantise simultaneously bosons and fermions and to exhibit the geometric origin of the so-called covariant derivatives. The paper is restricted to Galilean supersymmetry, but this is not a limitation of the procedure which applies to extended super-Poincaré supersymmetry as well.

1. Introduction

Supersymmetry appeared as a fundamental step towards generalising in a non-trivial and finite way the Poincaré space–time symmetry (for reviews see e.g. Corwin *et al* 1975, Fayet and Ferrara 1977, Salam and Strathdee 1978, van Nieuwenhuizen 1981), bypassing in this way the no-go theorems which had previously prevented such an extension (O'Raifeartaigh 1965a, b, Coleman and Mandula 1967). It has also stimulated the analysis of classical-mechanical systems described by a set of variables which includes Grassmann variables (pseudomechanics) (see e.g. Casalbuoni 1976a, b, Barducci *et al* 1976, Leites 1977) with the aim of reproducing the traditional structures of mechanics for systems with supersymmetry. Most of the analysis of supersymmetric theories has been performed in the framework of Einsteinian relativity, but sometimes the Galilean supersymmetry has also been considered (see de Franceschi and Palumbo 1980, Puzalowski 1978).

We have recently proposed (Aldaya and de Azcárraga 1982a, b, Aldaya *et al* 1982) a procedure of geometric quantisation† which is entirely based on the consideration of a group, the so-called 'quantum group'. The method, which incorporates many aspects of the Souriau–Kostant quantisation scheme (Souriau 1970, Kostant 1970; also see Lichnerowicz (1980) for another approach to quantisation), starts from a group $\tilde{G}$ which is a central extension of another group G by U(1). When this is applied to a free 'non-relativistic' system, $\tilde{G}$ turns out to be $\tilde{G}_{(m)}$, the eleven-parameter group obtained by adding a central generator to the Galilei group. This group may be viewed simply as the group obtained from G by substituting $[q^i, p_j] = m\delta^i_j I$ for

† A general outline of the method and its motivation is presented in 1983 *Proc. Int. Workshop on Supersymmetry*, ed B Milewski (to be published by World Scientific Publishing Co).

$[q^i, p_j] = 0$, a process which exhibits the special role of the mass in 'non-relativistic' quantum mechanics. It has been shown (Aldaya and de Azcárraga 1982a, b, Aldaya *et al* 1982) that, starting from $\tilde{G}_{(m)}$, it is possible to derive the quantum evolution equation and to define the quantum operators for a free Galilean particle; other interacting systems are considered as well. In this paper we extend our quantisation formalism to the case of the super-Galilei group. Once the differences of including Grassmann variables are properly taken into account, we obtain the result that the Galilean superfield incorporates simultaneously the description of Bose and Fermi particles distinguished by the statistics (and, consequently, by the action of the spin operator). In the process of reducing the superfield, the role of the 'covariant derivatives' (see e.g. Wess 1978, Zumino 1980, Corwin *et al* 1975, Fayet and Ferrara 1977, Salam and Strathdee 1978, van Nieuwenhuizen 1981) will be that of the polarisations of the geometric quantisation formalism† and its geometrical origin (as left invariant vector fields compatible with the action of the operators) will become apparent.

The analysis of this paper will be entirely 'non-relativistic' (i.e. Galilean) but some comments are in order here concerning the relativistic case. The group $\tilde{G}_{(m)}$ is a central extension of the Galilei group G by U(1); transforming it into a supergroup $\widetilde{SG}_{(m)}$ will involve the addition of the Grassmann variables, the supertranslations η_α, $\bar{\eta}_\alpha$ ($\alpha = 1, 2$) and a modification of the cocycle‡ defining the extension $\tilde{G}_{(m)}$ of G (Bargmann 1954) in order to have a non-trivial extension $\widetilde{SG}_{(m)}$ of the super-Galilei group SG by U(1). However, in the case of the Poincaré group no non-trivial extension by U(1) is possible. Nevertheless, it is known that the N super-Poincaré symmetry allows for non-trivial extensions by central charges (Haag *et al* 1978). Thus, it is remarkable that the U(1)-extended $N = 2$ super-Poincaré symmetry permits us to perform a quantisation in the Grassmann sector by the above procedure which is not allowed by the Poincaré group. In this way, Fermi and Bose statistics are required by special relativity. The extended $N = 2$ super-Poincaré group determines a superfield which incorporates two fields satisfying the Klein-Gordon equation and another one which verifies the Dirac equation (Aldaya and de Azcárraga 1983).

2. Quantisation and the super-Galilei group $\widetilde{SG}_{(m)}$

The quantisation procedure mentioned above starts with the quantum Lie group $\tilde{G}$, the central extension of G by U(1). On $\tilde{G}$ we may define the invariant left (right) vector fields $\mathscr{X}^{\mathrm{L}}(\tilde{G})$ ($\mathscr{X}^{\mathrm{R}}(\tilde{G})$); they verify that $\mathscr{X}^{\mathrm{L}}(\tilde{G}) \approx \mathscr{X}^{\mathrm{R}}(\tilde{G}) \approx \tilde{\mathscr{G}} = Te(\tilde{G})$ and $[\mathscr{X}^{\mathrm{L}}(\tilde{G}), \mathscr{X}^{\mathrm{R}}(\tilde{G})] = 0$. Another structure canonically defined on $\tilde{G}$ is the left (say) canonical one-form θ, which is a $\tilde{\mathscr{G}}$-valued one-form on $\tilde{G}$ defined by $\theta(X^{\mathrm{L}}_{\tilde{G}}) = X^{\mathrm{L}}_{\tilde{G}}$ $\forall X^{\mathrm{L}}_{\tilde{G}} \in \mathscr{X}^{\mathrm{L}}(\tilde{G})$. Defining a basis on $\tilde{\mathscr{G}}$, θ may be expressed as $\theta = \theta^{(i)\mathrm{L}} \circ X^{\mathrm{L}}_{\tilde{G}(i)}$ in terms

† Polarisations are introduced in the geometric quantisation approach to restrict the dependence of the wavefunctions on the variables of the theory. In this way the quantum operators adopt the customary (irreducible) form $\hat{X} = \mathrm{i}\,\partial/\partial p$, $\hat{p} = p$, and not $\hat{X} = \mathrm{i}\,\partial/\partial p + x$, $\hat{p} = -\mathrm{i}\,\partial/\partial x$ for instance. See e.g. Souriau (1970).

‡ The cocycle of the extension is a function $\xi: C \times G \to R$ which satisfies $\xi(g', g) + \xi(g'g, g'') = \xi(g', gg'') + \xi(g', g'')$, $\xi(g, e) = 0 = \xi(e, g)$; these properties guarantee that the composition law of the group $\tilde{G}$ of elements $(g \in G, \zeta \in \mathrm{U}(1))$, $(g', \zeta')(g, \zeta) = (g'g, \zeta'\zeta \exp \mathrm{i}\, \xi(g', g))$ is a group law. ξ is also called an exponent. Different ξ's differing in a coboundary define the same extension (see e.g. Bargmann (1954), where the case of the Galilei group is discussed in detail).

of ordinary (left) invariant one-forms $\theta^{(i)L}$. Since $\tilde{G}$ has a principal bundle structure $\tilde{G} \xrightarrow{\pi} \tilde{G}/U(1) \approx G$, the vertical component Θ of θ is canonically defined. Then ($P \equiv \tilde{G}/\mathscr{C}_\Theta$, $\Lambda = \Theta/\mathscr{C}_\Theta$), where $\mathscr{C}_\Theta$ is the characteristic module of Θ, is a quantum manifold; indeed, ($P \xrightarrow{\pi} P/U(1) \equiv S$, U(1)) is a principal bundle with Λ as connection one-form which defines on $SC\omega = \text{curv}\,\Lambda$) a symplectic structure. The characteristic module $\mathscr{C}_\Theta$ plays an important role in the formalism because it incorporates, in a wide sense, the equations of motion which are not immediately given in the traditional formalism. In our approach, the equations of motion will be determined by $\mathscr{C}_\Theta$ which is included as a polarisation, the polarisation being defined as a subspace of $\mathscr{X}^L(\tilde{G})$ (or its associated integral manifold) including $\mathscr{C}_\Theta$ and associated with a subalgebra of $\mathscr{X}^L(G)$. The wavefunctions are defined as functions $\Psi: \tilde{G} \to \mathbb{C}/\Psi(z * \tilde{g}) = z_{\mathbb{C}} * \Psi(\tilde{g})$; the (pre) quantum operators are obtained from the right vector fields $\mathscr{X}^R(\tilde{G})$ acting as derivations on the Ψ's and the full quantisation is obtained by imposing the polarisation conditions ($X^L \cdot \Psi = 0$, where X^L represent the vector fields defining the polarisation).

The above procedure of geometric quantisation carries forward to the case when $\tilde{G}$ is a supergroup $\widetilde{SG}$. Let us here consider the case of Galilean supersymmetry; for convenience we shall consider only one spatial dimension. The group law for the super-Galilei group $g' * g = g''$ is given by

$$B'' = B' + B \qquad A'' = A' + A + BV' \qquad V'' = V' + V \qquad \varphi'' = \varphi' + \varphi$$

$$\begin{aligned} \eta''^{\alpha} &= U(\varphi')^{\alpha}_{\ \beta}\eta^{\beta} + \eta'^{\alpha} \\ \bar{\eta}''^{\alpha} &= \bar{\eta}^{\beta} U^*(\varphi')_{\beta.}^{\ \ \alpha} + \bar{\eta}'^{\alpha} \end{aligned} \qquad U(\varphi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} = e^{i\sigma_3\varphi/2} \tag{2.1}$$

where A, B are the space–time translations, V the velocity and η^α, $\bar{\eta}^\alpha$ the supertranslations which provide a representation space for the $D^{1/2}$ representation ($\alpha = 1, 2$) of the rotation φ around the space axis. To obtain the U(1) extension one has to define a cocycle $\xi : (g', g) \mapsto \mathbb{R}$; once ξ is determined, the group law for $\widetilde{SG}$ will be given by ($\tilde{g} \equiv (g, \zeta)$, $\zeta \in U(1)$):

$$\tilde{g}'' = \tilde{g}' * \tilde{g} \qquad \tilde{g}'' = (g' * g, \zeta'\zeta \exp i\xi(g', g)). \tag{2.2}$$

It is not difficult to check that

$$\xi_{(m)}(g', g) = m\{A'V + B(V'V + \tfrac{1}{2}V'^2) + [\bar{\eta}'^{\gamma}\delta_{\gamma\alpha}U(\varphi')^{\alpha}_{.\beta}\eta^{\beta} - \bar{\eta}^{\gamma}\delta_{\gamma\beta}U(-\varphi')^{\beta}_{.\alpha}\eta'^{\alpha}]\}, \tag{2.3}$$

where the first part is a cocycle for $\tilde{G}_{(m)}$, fulfils the cocycle conditions for $SG_{(m)}$. It should be noted—apart from the appearance of the parameter m, to be identified with the mass—that the addition of a coboundary to the cocycle provides the same extension, but will modify the expression of the vector fields (see below) and the covariant derivatives. From the group law (2.1)–(2.3), and remembering the Grassmann character of the supertranslations, the left and right vector fields are easily found with the result

$\mathscr{X}^L(\widetilde{SG}_{(m)})$:	$\mathscr{X}^R(\widetilde{SG}_{(m)})$:
$X^L_B = \partial/\partial B + V\,\partial/\partial A + \frac{1}{2}mV^2$	$X^R_B = \partial/\partial B$
$X^L_A = \partial/\partial A$	$X^R_A = \partial/\partial A + mV\Xi$
$X^L_V = \partial/\partial V + mA\Xi$	$X^R_V = \partial/\partial V + B\,\partial/\partial A + mBV\Xi$

$$X^{\mathrm{L}}_{\varphi} = \partial/\partial\varphi \qquad X^{\mathrm{R}}_{\varphi} = \partial/\partial\varphi + \mathrm{i}\left(\frac{\sigma_3}{2}\right)^{\alpha}_{.\beta}\eta^{\beta}\frac{\partial}{\partial\eta^{\alpha}} - \bar{\eta}_{\beta}\mathrm{i}\left(\frac{\sigma_3}{2}\right)^{\beta}_{.\alpha}\frac{\partial}{\partial\bar{\eta}_{\alpha}}$$

$$Q^{\mathrm{L}}_{\alpha} = U(\varphi)^{\beta}_{.\alpha}[\partial/\partial\eta^{\beta} - m\bar{\eta}_{\beta}\Xi] \equiv U(\varphi)^{\beta}_{.\alpha}D^{\mathrm{L}}_{\beta} \qquad Q^{\mathrm{R}}_{\alpha} = \partial/\partial\eta^{\alpha} + m\bar{\eta}_{\alpha}\Xi \tag{2.4}$$

$$\bar{Q}^{\mathrm{L}\alpha} = U^{\dagger}(\varphi)^{\alpha}_{.\beta}[\partial/\partial\bar{\eta}_{\beta} - m\eta^{\beta}\Xi] \equiv U^{\dagger}(\varphi)^{\alpha}_{.\beta}\bar{D}^{\mathrm{L}\beta} \qquad \bar{Q}^{\mathrm{R}\alpha} = \partial/\partial\bar{\eta}_{\alpha} + m\eta^{\alpha}\Xi$$

$$\Xi \equiv \mathrm{i}\zeta\,\partial/\partial\zeta \qquad \Xi \equiv \mathrm{i}\zeta\,\partial/\partial\zeta$$

where Ξ is the central generator of $\widetilde{SG}_{(m)}$. The vertical component of the invariant canonical one-form on $\widetilde{SG}_{(m)}$ is immediately obtained from $\mathscr{X}^{\mathrm{L}}(\widetilde{SG}_{(m)})$ by imposing the conditions $\Theta(\Xi) = 1$, Θ (any other $X^{\mathrm{L}}) = 0$. The result is

$$\Theta = -mA\,\mathrm{d}V - \tfrac{1}{2}mV^2\,\mathrm{d}B + m\bar{\eta}_{\gamma}\,\mathrm{d}\eta^{\gamma} + m\eta^{\gamma}\,\mathrm{d}\bar{\eta}_{\gamma} + (\mathrm{d}\zeta/\mathrm{i}\zeta). \tag{2.5}$$

The explicit form of Θ depends on the election of the extension cocycle (2.3); nevertheless, the consequences will not depend on it (see the previous footnote).

It is clear that $(\widetilde{SG}_{(m)}, \Theta)$ is not a contact manifold†. However, we may obtain a contact manifold by taking the quotient over the characteristic module $\ker\Theta \cap \ker \mathrm{d}\Theta$. Again, taking care of the odd character of the η's it is easily found that the characteristic module is generated by the vector fields $X^{\mathrm{L}}_B, X^{\mathrm{L}}_{\varphi}$. The equations of motion corresponding to X^{L}_B are

$$\mathrm{d}B/\mathrm{d}t = 1 \qquad \mathrm{d}A/\mathrm{d}t = V \qquad \mathrm{d}V/\mathrm{d}t = 0 \qquad \mathrm{d}\zeta/\mathrm{i}\zeta = \tfrac{1}{2}mV^2\,\mathrm{d}t$$
$$\mathrm{d}\varphi/\mathrm{d}t = 0 \qquad \mathrm{d}\eta/\mathrm{d}t = \mathrm{d}\bar{\eta}/\mathrm{d}t = 0; \tag{2.6}$$

their solution is, trivially,

$$B = t \qquad A = (P/m)t + K \equiv x \qquad V = (P/m) \qquad \zeta = z\,\exp(\tfrac{1}{2}mV^2t)$$
$$\varphi = \varphi_0, \qquad \eta = \eta_0, \qquad \bar{\eta} = \bar{\eta}_0 \tag{2.7}$$

where $(K, P, \eta_0, \bar{\eta}_0, z)$ are the integration constants. In addition, the set of equations coming from X^{L}_{φ} simply tells us that there is no motion in the φ variable. Substituting (2.7) into (2.5) we may now obtain—if desired—the contact form $\Lambda = \Theta/\mathscr{C}_{\Theta}$

$$\Lambda = -K\,\mathrm{d}P + m\bar{\eta}_0\,\mathrm{d}\eta_0 - m\,\mathrm{d}\bar{\eta}_0\eta_0 + (\mathrm{d}z/\mathrm{i}z); \tag{2.8}$$

obviously $\mathrm{d}\Lambda$ is a symplectic form (note that it is symmetric in the Grassmann sector) and thus $(S, \mathrm{d}\Lambda)$, where $S \equiv [\widetilde{SG}_{(m)}/\mathscr{C}_{\Theta}]/\Xi$ is parametrised by $(K, P, \eta_0, \bar{\eta}_0)$, is a graded symplectic manifold‡.

† An *exact contact manifold* is the pair $(\mathscr{C}, \Lambda)$ where $\mathscr{C}$ is a manifold of dimension $2n+1$ and Λ is a one-form of constant class $2n+1$, i.e. such that $\ker\Lambda \cap \ker \mathrm{d}\Lambda \equiv \{X | i_X\Lambda = 0 = i_X\,\mathrm{d}\Lambda\} = 0$, i.e. has codimension $2n+1$.

‡ Graded symplectic manifolds in the context of prequantisation were first considered by Kostant (1975, in particular, §§ 5 and 6). Here we are not directly concerned with the problem of looking for the coadjoint orbits of a graded Lie group, but rather with the problem of quantisation by working directly on a 'quantum supergroup' which completely determines the process.

Following our formalism, the superwavefunction is a U(1)-function $\Psi:\widetilde{SG}_{(m)}\to\mathbb{C}$; thus $\Psi=\Psi(t,x,p,\varphi,\eta,\bar{\eta},\zeta)$ where the variables A and B have been redefined following (2.7). The U(1)-function condition $\Xi\cdot\Psi=\mathrm{i}\Psi$ implies that $\Psi=\tilde{\Phi}(t,x,p,\varphi,\eta,\bar{\eta})\zeta$. We now impose the polarisation (Planck) conditions to reduce $\tilde{\Phi}$; we take as polarisation the set of vector fields on $\widetilde{SG}_{(m)}$ $\{X_B^L, X_\varphi^L, X_A^L, \bar{Q}_\alpha^L\}$. Note that X_B^L will give rise to the evolution equation (wave equation) and that the condition $\bar{Q}_\alpha^L\cdot\tilde{\Phi}=0$ is nothing but the condition of null 'covariant derivative' used in supersymmetry for reducing the superfields†. Explicitly, $\bar{Q}_\alpha^L\cdot\tilde{\Phi}=0$ reads

$$(\partial\tilde{\Phi}/\partial\bar{\eta}^\alpha)-\mathrm{i}m\delta_{\alpha\beta}\eta^\beta\tilde{\Phi}=0\Rightarrow\tilde{\Phi}=\Phi(t,x,p,\varphi,\eta)\exp \mathrm{i}m\bar{\eta}^\alpha\eta^\beta\delta_{\alpha\beta}. \tag{2.9}$$

The superfield $\Phi(t,x,p,\varphi,\eta)\equiv\Phi(\varkappa,\eta)$ and the exponential may be expanded in finite powers of η with coeficients which are arbitrary functions of $\varkappa$

$$\Phi(\varkappa,\eta)=\mathfrak{A}(\varkappa)+\psi^\alpha(\varkappa)(\mathrm{i}\sigma_2)_{\alpha\beta}\eta^\beta+C(\varkappa)\eta^\beta(\mathrm{i}\sigma_2)_{\beta\alpha}\eta^\alpha$$

$$\exp \mathrm{i}m\bar{\eta}\eta=1+\mathrm{i}m\bar{\eta}\eta-\tfrac{1}{2}m^2(\bar{\eta}\eta)(\bar{\eta}\eta). \tag{2.10}$$

Thus

$$\tilde{\Phi}(\varkappa,\bar{\eta},\eta)=\mathfrak{A}(\varkappa)+\mathrm{i}m\mathfrak{A}(\varkappa)\bar{\eta}\eta-\tfrac{1}{2}m^2\mathfrak{A}(\varkappa)(\bar{\eta}\eta)(\bar{\eta}\eta)+\psi^\alpha(\varkappa)(\mathrm{i}\sigma_2)_{\alpha\beta}\eta^\beta$$

$$+\mathrm{i}m\psi^\alpha(\varkappa)(\mathrm{i}\sigma_2)_{\alpha\beta}\eta^\beta(\bar{\eta}\eta)+C(\varkappa)\eta^\alpha(\mathrm{i}\sigma_2)_{\alpha\beta}\eta^\beta. \tag{2.11}$$

The polarisation condition $X_A^L\cdot\tilde{\Phi}=0$ now eliminates the x dependence in all coefficients $\mathfrak{A}$, ψ^α and C; $X_\varphi^L\cdot\tilde{\Phi}=0$ eliminates the φ dependence. Thus the fields $\mathfrak{A}$, ψ^α and C which appear in (2.11) depend effectively only on (t,p). To obtain the equations of motion which are satisfied by these fields we have to impose the last polarisation condition, $X_B^L\cdot\Psi=0$. Recalling that $\Xi\cdot\Psi=\mathrm{i}\Psi$, one finds that the two types of fields involved satisfy the Schrödinger equation and the Schrödinger–Pauli equation respectively:

$$\mathrm{i}\frac{\partial\mathfrak{A}(t,p)}{\partial t}=\frac{p^2}{2m}\mathfrak{A}(t,p)\qquad \mathrm{i}\frac{\partial\psi^\alpha(t,p)}{\partial t}=\frac{p^2}{2m}\psi^\alpha(t,p) \tag{2.12}$$

and that the final form of Ψ is given by the expression

$$\Psi=[\mathfrak{A}(p)+\psi^\alpha(p)(\mathrm{i}\sigma_2)_{\alpha\beta}\eta^\beta+C(p)\eta^\beta(\mathrm{i}\sigma_2)_{\beta\alpha}\eta^\alpha]\zeta\,(\exp-\mathrm{i}(p^2/2m)t)\,(\exp \mathrm{i}m\bar{\eta}\eta)$$

$$\equiv\Phi(p,\eta)\zeta\,(\exp-\mathrm{i}(p^2/2m)t)(\exp \mathrm{i}m\bar{\eta}\eta). \tag{2.13}$$

The quantum operators are, according to the general formalism (see Aldaya and de Azcárraga 1982, Aldaya *et al* 1982), the right vector fields (apart from a numerical factor) acting as derivations on the polarised wavefunctions. It should be noted that, because right and left vector fields commute, the action of the operators is compatible with all the restrictions (polarisations) introduced. We take

$$\hat{K}\equiv\frac{\mathrm{i}}{m}X_V^R\qquad \hat{P}\equiv-\mathrm{i}X_A^R\qquad \hat{L}_z\equiv-\mathrm{i}X_\varphi^R\qquad \hat{H}\equiv\mathrm{i}X_B^R$$

$$\hat{Q}_\alpha\equiv\mathrm{i}Q_\alpha^R\qquad \hat{\bar{Q}}_\alpha\equiv-\mathrm{i}\bar{Q}_\alpha^R. \tag{2.14}$$

† More precisely the 'covariant derivatives' (see e.g. Salam and Strathdee (1978) for the relativistic case) are just the D^L, $\bar{D}^L$ of (2.4). Because $U(\varphi)$ $U^+(\varphi)$ are regular matrices, they provide the same constraints. The Q^L; $\bar{Q}^L$ are 'invariant' (commute) with the right-invariant vector fields and in particular with the rotations generator X_φ^R. It should be noted that the above 'superfield' Ψ depends on more variables than the superfield of Salam and Strathdee which only depends on the superspace variables.

Eliminating the factor ζ, one obtains

$$\hat{K}\tilde{\Phi} = (\mathrm{i}\,\partial/\partial p - tp/m)\Phi, \tag{2.15}$$

and eliminating the exponential we get for the Bose (i.e. $\mathfrak{A}$ and C) and Fermi (ψ^α) fields contained in the superfield

$$\hat{K}\mathfrak{A}(p) = \mathrm{i}\,(\partial/\partial p)\mathfrak{A}(p) \qquad \hat{K}\psi^\alpha(p) = \mathrm{i}\,(\partial/\partial p)\psi^\alpha(p). \tag{2.16}$$

(2.15) shows that in momentum representation, $\hat{x} = \mathrm{i}(\partial/\partial p)$ (we have put $\hbar = 1$ throughout).

The operator $\hat{L}_z$ simply confirms the scalar and spinorial character of $\mathfrak{A}$, C and ψ^α respectively. Indeed, taking into account that $X^{\mathrm{R}}_{\varphi} \cdot \eta(\mathrm{i}\sigma_2)\eta = 0$, $X^{\mathrm{R}}_{\varphi} \cdot \bar{\eta}\eta = 0$, one obtains that the induced action of $\hat{L}_z$ on the fields contained in Ψ is given by

$$\hat{L}_z\mathfrak{A} = 0 \qquad \hat{L}_z C = 0 \qquad \hat{L}_z\psi^\alpha = \tfrac{1}{2}(\sigma_3)^{\alpha}_{\cdot\beta}\psi^\beta. \tag{2.17}$$

Thus, the superfield Ψ contains two scalar fields and one fermion field described by a Pauli spinor $\psi^\alpha(t, p)$.

Finally, the operators $\hat{Q}_\alpha$ and $\hat{\bar{Q}}_\alpha$ play the role of canonical conjugate operators, their actions on $\Phi(t, p, \eta)$ being given by

$$\hat{Q}_\alpha\Phi = \mathrm{i}(\partial/\partial\eta^\alpha)\Phi \qquad \hat{\bar{Q}}_\alpha\Phi = 2m\eta^\beta\delta_{\beta\alpha}\Phi \tag{2.18}$$

in accord with the fact that $\{Q^{\mathrm{R}}_\alpha, Q^{\mathrm{R}}_\beta\} = 2m\delta_{\alpha\beta}\Xi$. $\hat{P}$ and $\hat{H}$ are associated, as expected, with the momentum and the energy.

To conclude, we remark the crucial role played by the mass m in the definition of the extensions $\tilde{G}_{(m)}$ and $\widetilde{SG}_{(m)}$. In the case of the extended $N = 2$ super-Poincaré symmetry the mass also appears associated with a central charge†. As mentioned in the introduction, it has been shown elsewhere that m characterises the cocycle defining the extended $N = 2$ super-Poincaré symmetry and that the above quantisation formalism leads to two Klein–Gordon and one Dirac equations.

Acknowledgments

This paper has been partially supported through a grant from the Comisión Asesora para la Investigación Científica y Técnica (contract 1261-82).

References

Aldaya V and de Azcárraga J A 1982a *Lett. Nuovo Cimento* **33** 341
—— 1982b *J. Math. Phys.* **23** 1297
—— 1983 *Phys. Lett.* **121B** 331
Aldaya V, de Azcárraga J A and Wolf K B 1982 *Physica* **114A** 468
—— 1983 *J. Math. Phys.* to appear
de Azcárraga J A and Lukierski J 1982 *Phys. Lett.* **113B** 170
Barducci A, Casalbuoni R and Lusanna L 1976 *Nuovo Cimento* **35A** 377
Bargmann V 1954 *Ann. Math.* **59** 1

† For a comparison of the Galilean and Poincaré cases in the presence of central charges in a pseudoclassical particle model see de Azcárraga and Lukierski (1982).

Casalbuoni R 1976a *Nuovo Cimento* **33A** 339 115
—— 1976b *Phys. Lett.* **62B** 49
Coleman S and Mandula J 1967 *Phys. Rev.* **159** 1251
Corwin L, Ne'eman Y and Sternberg S 1975 *Rev. Mod. Phys.* **47** 573
Fayet P and Ferrara S 1977 *Phys. Rep.* **32C** 249
de Franceschi G and Palumbo F 1980 *Nucl. Phys.* B **162** 478
Haag R, Łopuszański J T and Sohnius M 1978 *Nucl. Phys.* B **88** 257
Kostant B 1970 *Lect. Notes Math.* **170** 87
—— 1975 *Lect. Notes Math.* **570** 177
Leites D A 1977 *Dokl. Akad. Nauk* **18** 1277
Lichnerowicz A 1980 *Lect. Notes Math.* **836** 366
van Nieuwenhuizen P 1981 *Phys. Rep.* **68C** 189
O'Raifeartaigh L 1965a *Phys. Rev. Lett.* **14** 575
—— 1965b *Phys. Rev.* **139B** 1052
Puzalowski R 1978 *Acta Phys. Austriaca* **50** 45
Salam S and Strathdee 1978 *Fortschr. Phys.* **26** 57
Souriau J M 1970 *Structure des Systèmes Dynamiques* (Paris: Dunod)
Wess J 1978 *Topics in Quantum Field Theory and Gauge Theories* ed J A de Azcárraga (Berlin: Springer)
Zumino B 1980 *Superspace, Erice Workshop on the Fundamental Interactions, CERN preprint Th.2852*